Received: *June 6, 2000*, Accepted: *August 3, 2000*
Revised: *August 2, 2000*

Flavour stability of the chiral vacuum and scalar meson dynamics

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Abstract: Previous work relating the flavour variation of the chiral order parameters F_π, $\langle \bar{u}u \rangle$ and S-wave scattering data, based on chiral sum rules and chiral perturbation theory at order p^4, is extended to include $O(p^6)$ corrections. The finding of a significant decrease of these order parameters, particularly $\langle \bar{u}u \rangle$, with the number of flavours increasing from $N_F = 2$ to $N_F = 3$ is confirmed, modulo an assumption on the convergence of the chiral expansion. The connection between scalar resonance physics and the phase structure of the chiral vacuum is also illustrated on the basis of the linear sigma-model. We allow for a very general symmetry breaking sector compatible with softness. The result depends strongly on the input scalar meson masses, in particular, on the presence, or not, of a light sigma.

Keywords: Chiral Lagrangians, Sum Rules, Phenomenological Models.

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1. Introduction

Chiral perturbation theory for $N_F = 3$ light flavours at next-to-leading order was shown by Gasser and Leutwyler [1] to involve ten independent coupling constants $L_i^r(\mu)$ (referred to, sometimes, as LEC's). In ref. [1] all ten couplings were determined from low-energy data except two, L_4 and L_6. According to the OZI rule or, alternatively, large-N_c considerations these couplings are expected to be suppressed relative to the other ones [1]. Physically, L_4 and L_6 control how observables of the pion, like M_π or chiral SU(2) order parameters $(F_\pi)_{\mathrm{SU}(2)}$, $\langle \bar{u}u \rangle_{\mathrm{SU}(2)}$ vary if the mass of the strange quark mass m_s varies. If OZI suppression holds these quantities are expected to be essentially insensitive to variations of m_s and the value of the chiral order parameters F_π, $\langle \bar{u}u \rangle$ in the SU(2) chiral limit or in the SU(3) chiral limit (i.e. $m_s = 0$) ought to be very nearly the same. In nature, of course, the values of the light quark masses are fixed but they can be made to vary in unquenched lattice simulations of QCD.

There are reasons to suspect that the couplings L_4 and L_6 are, in fact, far from being suppressed. One reason is suggested by two recent unquenched lattice simulations which have investigated the phase structure of QCD-like theories with

N_F equal-mass flavours when the value of N_F is varied. It was found in ref. [2] that for $N_F > 6$ a phase with no confinement and no spontaneous chiral symmetry breaking prevails, at any value of the QCD coupling constant. In ref. [3] a very strong decrease of the chiral order parameters was observed upon varying N_F from $N_F = 2$ to $N_F = 4$. The values of the coupling constants L_i encode informations on the physics of the massive states in QCD [4]. In particular, L_4, L_6 are linked to the scalar resonances. It is notorious that the OZI or large-N_c rules seem to fail in this sector. In fact, is not clear at present exactly which of the scalar resonances form the lowest lying nonet (see e.g. the reviews in the pdg [5]). In other words, the fact that L_4, L_6 may be unsuppressed, the failure of the OZI rule in the scalar sector are connected and these features may be related in an interesting way to the phase structure of QCD, in particular to the fact that a phase transition could occur for a value of N_F not exceedingly larger than $N_F = 3$. An interpretation of these features in terms of a paramagnetic effect of the quark loops is discussed in ref. [6].

These topics have started to be investigated in a previous paper [7]. The couplings L_4, L_6 were related to the scalar resonances via the scalar form-factors of the pion and the Kaon. These form-factors can be reconstructed directly from experimental data on S-wave scattering, modulo a few plausible hypothesis [8]. The coupling L_4 is related to the derivative of the strange form-factor of the pion at the origin and, for L_6, a chiral sum rule can be derived in the terms of the correlator Π_6 of the strange scalar current $\bar{s}s$ and the non-strange current $\bar{u}u + \bar{d}d$. In the present paper, we investigate the $O(m_s)$ corrections to the results of ref. [7], which could be sizable. For this purpose, one needs to use CHPT at order p^6. Renormalizability of the theory at this order was recently proved [9], and the set of independent chiral lagrangian terms classified [10]. In particular, we will use the expansions of M_π and F_π which were derived in refs. [11, 12] and we have computed the $O(p^6)$ expansion of the correlator Π_6. The result of this effort, at first sight, will turn out to be mitigated: as new coupling constants appear in the formulas, there is a loss in predictivity over the leading order calculation. As discussed in section 2, conclusions may nevertheless be drawn provided one makes some assumptions on the convergence of the chiral expansion.

In order to gain further insight, we have also investigated a different, more model dependent approach in section 3. Following the idea of ref. [4] one starts from a lagrangian for the scalar resonances and the LEC's are generated by integrating out the resonances. If we attempt to determine some $O(p^6)$ LEC's in this way, from the most general lagrangian, we find that there are many undetermined resonance parameters which get involved, like three resonance couplings, couplings of one resonance to two scalar sources etc. In section 3, we discuss the more specific dynamics generated by the linear sigma-model [13]. This model is still attracting interest [14, 15] in connection with the phenomenon of disordered chiral condensate [16] which could be formed in heavy ion collisions. In the symmetry breaking sector, we will consider a lagrangian more general than previously done, still compatible with the criterion of softness (restricting ourselves to first order symmetry breaking). The model can accommodate exactly, in principle, a given set of scalar nonet masses and one obtains the $O(p^4)$ and $O(p^6)$ LEC's as definite predictions. The question at this point is which scalar mesons are to be included in the nonet? In particular, does one have to include a light σ and a light κ meson? It has long been known that an S-matrix pole can be identified in low-energy $\pi\pi$ scattering [18] with a very large imaginary part. A similar structure was argued to exist also in πK scattering [19, 20], but it is unclear whether such poles should be interpreted as physical scalar resonances. For our purposes, we will leave this question open and consider two different possibilities for the scalar nonet. As we will see, in the framework of the sigma-model, these will correspond to rather different behaviour of the chiral vacuum.

2. L_4, L_6 beyond $O(p^4)$

Our estimates of L_4 and L_6 are based, essentially, on a method to extract the scalar form-factors of the pion and the kaon using experimental $\pi\pi - K\bar{K}$ scattering data, proposed in ref. [8]. The form-factors get determined up to a normalization factor, for which one uses CHPT. In order to determine L_4 one equals the experimental determination of $G'_\pi(0)$, the derivative of the strange scalar form-factor of the pion and the chiral expansion of this quantity. In ref. [7] this matching was performed at leading order in the chiral expansion, i.e. $G'_\pi(0)$ was expanded up to $O(p^4)$ and for the normalization condition we used CHPT at order p^2. In a similar way, for L_6 we use the scalar form factors to express the spectral function of the correlator $\Pi_6(s)$ (see (2.13) below) and L_6 is obtained by matching $\Pi_6(0)$ evaluated from experimental data and its chiral expansion. Below, we discuss the chiral corrections to these results, which involve two parts (a) one must use the chiral expansions of $G'_\pi(0)$ and $\Pi_6(0)$ up to $O(p^6)$ and (b) we must use CHPT at $O(p^4)$ instead of $O(p^2)$ in the normalization conditions.

Let us designate by $F_\pi(s)$ and $G_\pi(s)$ the non strange and strange scalar form factors of the pion, and by $F_K(s)$ and $G_K(s)$ the analogous ones for the Kaon. We will use the following normalizations,

$$\begin{aligned} F_\pi &= \frac{1}{B_0}\sqrt{\frac{3}{2}}\langle 0|\bar{u}u+\bar{d}d|\pi^0\pi^0\rangle\,, \qquad & G_\pi &= \frac{1}{B_0}\sqrt{\frac{3}{2}}\langle 0|\bar{s}s|\pi^0\pi^0\rangle\,, \\ F_K &= \frac{1}{B_0}\sqrt{2}\langle 0|\bar{u}u+\bar{d}d|K^+K^-\rangle\,, \qquad & G_K &= \frac{1}{B_0}\sqrt{2}\langle 0|\bar{s}s|K^+K^-\rangle\,. \end{aligned} \tag{2.1}$$

The method of ref. [8] consists in solving numerically a set of Muskhelishvili-Omnès coupled-channel equations using experimentally determined $\pi\pi - K\bar{K}$ S-wave scattering T-matrix elements as input(in practice we made use of the parametrisations of refs. [21, 22]). Strictly speaking, these equations hold under the assumption of exact two-channel unitarity up to $s=\infty$ while in practice, two-channel unitarity is a good approximation up to the $\eta\eta$ threshold. The resulting form factors are expected to be reliable in a finite energy range, which we will assume to extend up to 1 GeV. One also chooses appropriate boundary conditions for the T-matrix at $s=\infty$ which insure existence of a solution with a minimal number of free parameters: a solution vector $u_1(s),\ u_2(s)$ will be determined in the entire energy range once the values at one point, say at $s=0$, are given. Extension to more channels and the stability of this scheme were discussed in ref. [7]. In practice, one first constructs numerically two independent solutions of the equation set: $u_i(s)$ and $v_i(s)$, $i=1,2$, normalized at the origin such that $u_1(0)=1$, $u_2(0)=0$ and $v_1(0)=0$, $v_2(0)=1$. The form factors are then given as

$$\begin{pmatrix} F_\pi(s) \\ F_K(s) \end{pmatrix} = F_\pi(0) \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix} + F_K(0) \begin{pmatrix} v_1(s) \\ v_2(s) \end{pmatrix} \tag{2.2}$$

and similarly for the strange form factors. In refs. [8, 7] the normalization at the origin was taken from CHPT at $O(p^2)$. Here, we wish to investigate the chiral corrections to these results, so we need to go to the next chiral order.

2.1 Evolution of F_π

In the SU(2) chiral limit, the strange form factor of the pion vanishes at the origin $G_\pi(0)=0$, the LEC L_4 is related to the first derivative at the origin of G_π. It is convenient to consider the proportional quantity

$$d_F^{\exp} = \sqrt{\frac{2}{3}} m_s B_0\, G'_\pi(0) \equiv \sqrt{\frac{2}{3}} m_s B_0\, G_K(0) v'_1(0)\,. \tag{2.3}$$

Here, $v'_1(0)$ can be determined from experiment by the procedure outlined above, and one finds,

$$v'_1(0) \simeq 0.27\,\mathrm{GeV}^{-2}\,, \tag{2.4}$$

using the T-matrix parametrisation of ref. [22] (using that of ref. [21] one would find $v'_1(0) \simeq 0.31\,\mathrm{GeV}^{-2}$). We next need to express $m_s B_0$ and $G_K(0)$ in eq. (2.3). At leading chiral order, one has $m_s B_0 = m_K^2 - m_\pi^2/2$ and $G_K(0)=\sqrt{2}$. Including $O(m_s)$ corrections to this result, one obtains $d_F^{\exp}$ in the form,

$$d_F^{\exp} = \frac{2}{\sqrt{3}} \left[m_K^2 - \frac{m_\pi^2}{2} + \frac{m_K^4}{F_\pi^2} \left(-8S_8 - 16S_6 + \frac{1}{36\pi^2}(L_\eta + 1) \right) \right] v'_1(0)\,, \tag{2.5}$$

with

$$S_8 = -2L_8^r(\mu) + L_5^r(\mu)\,, \qquad S_6 = -2L_6^r(\mu) + L_4^r(\mu)\,, \qquad L_\eta = \log\frac{m_\eta^2}{\mu^2}\,. \tag{2.6}$$

Corrections of order $m_\pi^2 m_K^2$ or m_π^4 have been neglected here and CHPT at $O(p^4)$ has been used to obtain $G_K(0)$ and to express $m_s B_0$ in terms of the physical Kaon mass.

We observe that low-energy constants appear now but it is consistent to use their values obtained at $O(p^4)$, i.e. for L_5, L_8 the values given in ref. [1] and for L_4, L_6 those obtained in ref. [7]. We can then match $d_F^{\rm exp}$ with its expression as a chiral expansion. The latter is obtained from the relation [8]

$$d_F = \frac{m_s}{F_\pi}\frac{dF_\pi}{dm_s} \tag{2.7}$$

which holds in the chiral SU(2) limit, $m_u = m_d = 0$, and one can use the chiral expansion of F_π which has been determined up to $O(p^6)$ [11, 12]. One obtains

$$d_F = \frac{m_s B_0}{F_0^2} d^{(4)} + \frac{m_s^2 B_0^2}{F_0^4} d^{(6)} + O(m_s^3)\,, \tag{2.8}$$

with

$$d^{(4)} = 8L_4^r - \frac{1}{32\pi^2}(L_K + 1)\,, \quad L_K = \log\frac{m_s B_0}{\mu^2}\,, \tag{2.9}$$

and

$$\begin{aligned} d^{(6)} &= 64C_{16}^r - 128(L_4^r)^2 - \frac{17}{3072\pi^4}L_K^2 - \frac{5}{1152\pi^4}L_K L_\eta + \\ &+ \frac{L_K}{\pi^2}\left(\frac{73}{9216\pi^2} + 4L_1^r + L_2^r + \frac{5}{4}L_3^r + \frac{1}{2}L_5^r - 2L_6^r - L_8^r\right) + \\ &+ \frac{L_\eta}{\pi^2}\left(-\frac{1}{768\pi^2} + \frac{16}{9}L_1^r + \frac{4}{9}L_2^r + \frac{4}{9}L_3^r - \frac{8}{9}L_4^r\right) + \\ &+ \frac{1}{\pi^2}\left(\frac{26}{9}L_1^r + \frac{35}{54}L_3^r + \frac{5}{9}L_4^r + \frac{3}{4}L_5^r - 3L_6^r - \frac{3}{2}L_8^r - \frac{0.0022}{\pi^2}\right). \end{aligned} \tag{2.10}$$

In this formula, the finite contribution from the so-called sunset diagrams has been evaluated numerically. Equating (2.3) and (2.8) at the leading, linear order in m_s, gives the $O(p^4)$ determination of L_4. From the numerical results of refs. [8, 7] one obtains

$$L_4^r(m_\eta) \simeq 0.6\,10^{-3} \quad [\text{order } p^4]\,. \tag{2.11}$$

If we include the corrections quadratic in m_s now, a new $O(p^6)$ coupling constant appears, labelled C_{16} in ref. [10], so strictly speaking, we have one equation and two unknowns. A similar situation will prevail in the determination of L_6 to be discussed below. We expect, however, the chiral expansion to be meaningful and the $O(p^6)$ part of the expansion to be smaller than the $O(p^4)$ one. If we assume a given ratio for these two parts we can determine L_4 and C_{16} separately. Once these two couplings are known we can, furthermore, using the CHPT expressions of refs. [11, 12], determine how F_π varies in going from an SU(2) chiral limit to an SU(3) one. Some results are collected in table 1, where we have varied the $O(p^6)$ over $O(p^4)$ ratio between 10% and 100%.

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$O(p^6)/O(p^4)$	$10^3\,L_4$	$10^5\,C_{16}$	F/F_0-1
0.10	0.58 [0.31]	0.36 [0.24]	$0.18+0.005$
0.30	0.53 [0.26]	0.39 [0.26]	$0.17+0.011$
0.50	0.49 [0.22]	0.41 [0.28]	$0.16+0.016$
1	0.42 [0.15]	0.44 [0.32]	$0.14+0.024$

Table 1: Assuming a given ratio of the $O(p^6)$ to the $O(p^4)$ contributions in the chiral expansion of d_F, the values of the LEC's $L_4^r(\mu)$ and $C_{16}^r(\mu)$ are computed for $\mu = m_\eta$ (in brackets, $\mu = m_\rho$). We also display the successive contributions to the ratio of $F \equiv (F_\pi)_{\mathrm{SU}(2)}$ over $F_0 \equiv (F_\pi)_{\mathrm{SU}(3)}$.

This exercise seems to indicate that the determination of L_4 is rather stable and rather close to its determination at leading order, and provides an estimate for the size of the $O(p^6)$ coupling constant C_{16}^r. A priori, however, one cannot completely exclude a different solution, with $L_4(m_\eta) = 0$, for instance. In this case, all the OZI violation would be concentrated in the $O(p^6)$ parameter, which seems somewhat unplausible. Taking into account the experimental uncertainties in the T-matrix, the uncertainty from the energy region above 1 GeV (which were estimated in ref. [7]) and the discussion above of the $O(m_s)$ corrections, we find that the error on L_4 should be of the order of 30%, i.e.

$$L_4(m_\eta) = (0.6 \pm 0.2)\, 10^{-3}\,. \tag{2.12}$$

2.2 Evolution of $\langle \bar{u}u \rangle$

Let us now discuss the coupling constant L_6. Using the same scalar form factors, it was remarked in ref. [7] that one can derive an estimate of L_6. The idea is to consider the two-point correlation function,

$$\Pi_6(p^2) = \frac{i}{B_0^2} \int d^4x\, e^{ipx} \langle 0|T(\bar{u}u(x) + \bar{d}d(x))\bar{s}s(0)|0\rangle_{conn.}\,. \tag{2.13}$$

In an energy domain extending roughly up to the $\eta\eta$ threshold, the spectral function is given in terms of the scalar form factors of the pion and the Kaon,

$$\begin{aligned} \mathrm{Im}\,\Pi_6(s) = & \sqrt{\frac{s-4m_\pi^2}{s}} F_\pi(s) G_\pi^*(s) \theta(s-4m_\pi^2) + \\ & + \sqrt{\frac{s-4m_K^2}{s}} F_K(s) G_K^*(s) \theta(s-4m_K^2)\,. \end{aligned} \tag{2.14}$$

Essentially, a usual assumption is made there, that the 4π or 6π contributions are negligibly small in this energy region (see, e.g. [21]). In the SU(2) chiral limit,

$m_u = m_d = 0$, a super-convergence relation holds,

$$\int_0^\infty \mathrm{Im}\,\Pi_6(s)\,ds = 0\,. \tag{2.15}$$

Similar chiral sum rules are known to be saturated to a fairly good approximation in terms of a few low-lying resonances (see, e.g. [23]). In the energy region below 1 GeV, inserting the form factors constructed as discussed above, the contribution to the integral is found to be positive, and dominated by the $f_0(980)$ resonance. A plausible assumption, then, is that the higher-energy contribution to the sum rule (2.15) will be saturated, at least approximately, by the next prominent scalar resonance, the $f_0(1500)$, which we expect to make a negative contribution. We observe that the signs of the contributions of the $f_0(980)$ and the $f_0(1500)$ conforms with the recent assignments of Minkowski and Ochs [24] as (essentially) $\mathrm{SU}_F(3)$ singlet and octet, respectively. Finally, one can calculate the value of the correlation function at zero energy, $\Pi_6(0)$, making use of the scalar form factors below 1 GeV, and the super-convergence relation (2.15) as,

$$\Pi_6(0) \simeq \frac{1}{\pi}\left(\int_0^1 \frac{\mathrm{Im}\,\Pi_6(s)}{s}\,ds - \frac{1}{s_0}\int_0^1 \mathrm{Im}\,\Pi_6(s)\,ds\right), \tag{2.16}$$

with $s_0 \simeq 1.5^2\,\mathrm{GeV}^2$. This gives us an "experimental" value of $\Pi_6(0)$, the difference with ref. [7] is that we now wish to take into account chiral corrections in the normalization of the form factors. Below the $K\bar{K}$ threshold, we can express the spectral function in terms of the basis solutions to the Muskhelishvili-Omnès equations $u_i(s)$, $v_i(s)$ as,

$$\mathrm{Im}\,\Pi_6(s) = \sqrt{\frac{s-4m_\pi^2}{s}}\left[F_\pi(0)G_K(0)u_1(s)v_1^*(s) + F_K(0)G_K(0)v_1(s)v_1^*(s)\right]. \tag{2.17}$$

Including $O(m_s)$ chiral corrections in the normalizing factors, we now have

$$\begin{aligned} F_\pi(0)G_K(0) &= 2\sqrt{3}\left(1+\frac{m_K^2}{F_\pi^2}\left[-16S_8 - 48S_6 + \frac{1}{24\pi^2}L_\eta + \frac{1}{36\pi^2}\right]\right), \\ F_K(0)G_K(0) &= 2\left(1+\frac{m_K^2}{F_\pi^2}\left[-32S_8 - 80S_6 + \frac{7}{72\pi^2}L_\eta + \frac{1}{24\pi^2}\right]\right), \end{aligned} \tag{2.18}$$

where $S_8 = -2L_8^r + L_5^r$, $S_6 = -2L_6^r + L_4^r$ and we may use in this part the values of the LEC's as determined at $O(p^4)$.

We then use the experimental result for $\Pi_6(0)$ in conjunction with its chiral expansion. The computation of $\Pi_6(s)$ to $O(p^6)$ is exposed in the appendix. Concerning $\Pi_6(0)$, the chiral expansion goes as follows:

$$\Pi_6(0) = \Pi^{(4)} + \frac{m_s B_0}{F_0^2}\Pi^{(6)}\,, \tag{2.19}$$

with

$$\Pi^{(4)} = 64L_6^r - \frac{1}{16\pi^2}\left(\frac{22}{9}(L_K+1)+\frac{4}{9}L_{43}\right), \tag{2.20}$$

and

$$L_K = \log\frac{m_s B_0}{\mu^2}, \qquad L_{43} = \log\frac{4}{3}. \tag{2.21}$$

This expression generates the $O(p^4)$ determination [7] of L_6,

$$L_6^r(m_\eta) \simeq 0.5\,10^{-3} \quad \left[\text{order } p^4\right]. \tag{2.22}$$

The expression of the $O(p^6)$ part in (2.19) is,

$$\begin{aligned}\Pi^{(6)} =& 256(C_{20}^r + 3C_{21}^r) - \frac{1}{72\pi^4}L_K^2 - \frac{1}{96\pi^4}L_K L_{43} + \frac{1}{288\pi^4}L_{43}^2 + \\ &+ \frac{L_K}{\pi^2}\left(\frac{64}{3}S_6 + \frac{70}{9}S_8 + \frac{16}{9}S_7 - \frac{55}{2592\pi^2}\right) + \\ &+ \frac{L_{43}}{\pi^2}\left(\frac{16}{3}S_6 + \frac{16}{9}S_8 + \frac{16}{9}S_7 - \frac{47}{5184\pi^2}\right) + \\ &+ \frac{1}{\pi^2}\left(\frac{140}{9}S_6 + \frac{175}{27}S_8 + \frac{8}{27}S_7 - \frac{19}{5184\pi^2}\right).\end{aligned} \tag{2.23}$$

with

$$S_7 = 3L_7 + L_8^r. \tag{2.24}$$

In these formulas, we have set $m_u = m_d = 0$. These expressions exhibit the explicit m_s dependence (which will prove usefull below as we plan to integrate over m_s). For numerical application, we may use $m_s B_0 = m_K^2 - m_\pi^2/2$ in $\Pi^{(6)}$ while in $\Pi^{(4)}$ we have to use the $O(p^4)$ expansions of m_K^2 and m_π^2 [1]. Equating the chiral expansion of $\Pi_6(0)$ with its sum rule evaluation gives us a linear relation involving the $O(p^4)$ LEC L_6 and the $O(p^6)$ combination $C_{20} + 3C_{21}$. As before, we must make a hypothesis concerning the convergence of the chiral expansion if we want to separately estimate these two contributions.

Another interesting information contained in the correlation function Π_6 concerns the variation of the quark condensate in the chiral SU(2) limit as a function of the strange quark mass:

$$\frac{d\langle\bar{u}u\rangle}{dm_s} = -\frac{B_0^2}{2}\Pi_6(0). \tag{2.25}$$

Using the chiral expansion of $\Pi_6(0)$ we can integrate this equation in the variable m_s from its physical value down to $m_s = 0$. In this way we obtain an estimate for how the condensate varies from a chiral SU(2) limit to a chiral SU(3) limit. This variation is obtained as an expansion in powers of the physical strange quark mass (or, alternatively, in powers of the Kaon mass),

$$\langle\bar{u}u\rangle_{\mathrm{SU}(2)} = \langle\bar{u}u\rangle_{\mathrm{SU}(3)}\left(1 + \frac{m_s B_0}{F_0^2}R^{(4)} + \frac{m_s^2 B_0^2}{F_0^4}R^{(6)}\right), \tag{2.26}$$

with

$$R^{(4)} = 32L_6^r - \frac{11}{144\pi^2}L_K - \frac{1}{72\pi^2}L_{43}\,, \tag{2.27}$$

and

$$\begin{aligned} R^{(6)} = {} & 64(C_{20}^r + 3C_{21}^r) - \frac{1}{288\pi^4}L_K^2 - \frac{1}{384\pi^4}L_K L_{43} + \frac{1}{1152\pi^4}L_{43}^2 + \\ & + \frac{L_K}{\pi^2}\left(\frac{16}{3}S_6 + \frac{4}{9}S_7 + \frac{35}{18}S_8 - \frac{19}{10368\pi^2}\right) + \\ & + \frac{L_{43}}{\pi^2}\left(\frac{4}{3}S_6 + \frac{4}{9}S_7 + \frac{4}{9}S_8 - \frac{5}{5184\pi^2}\right) + \\ & + \frac{1}{\pi^2}\left(\frac{11}{9}S_6 - \frac{4}{27}S_7 + \frac{35}{54}S_8\right). \end{aligned} \tag{2.28}$$

Using the relation

$$-2\langle \bar{u}u\rangle_{\mathrm{SU}(2)} = \frac{d}{dm}\left(m_\pi^2 F_\pi^2\right)_{m=0}\,, \qquad m = \frac{1}{2}(m_u + m_d)\,, \tag{2.29}$$

one sees that the expansion of the condensate can be rederived from the expansion of the product $m_\pi^2 F_\pi^2$ in powers of m_s. We have verified, using the $O(p^6)$ expansion [11, 12] of m_π, F_π that the formulas (2.26), (2.27) and (2.28) are exactly reproduced.

Let us now discuss some numerical results. At first, it is instructive to compare the value obtained for $\Pi_6(0)$ using the normalization of the form factors at $O(p^2)$ and the normalization at $O(p^4)$,

$$\Pi_6(0) \simeq 0.022 \quad \left[O(p^2)\ \text{norm.}\right], \qquad \Pi_6(0) \simeq 0.043 \quad \left[O(p^4)\ \text{norm.}\right]. \tag{2.30}$$

We see that including $O(m_s)$ corrections in the normalizations brings a rather substantial change in the result by approximately a factor of two. This can be traced to the large numbers appearing in front of the combination $S_6 = -2L_6 + L_4$ in the normalization factors (2.18). Evidently, the difference $-2L_6 + L_4$ is very sensitive to a small variation of L_6 or L_4. Taking L_6 slightly smaller than the central value (2.22) would decrease significantly the modification in $\Pi_6(0)$ without essentially altering the other results. Upon considering the chiral expansion of $\Pi_6(0)$ now, this result suggests that the $O(p^6)$ contribution could be substantial and it is likely to be positive. We have collected some results in table 2 assuming that the ratio of the $O(p^6)$ to the $O(p^4)$ contributions is positive and ranges from 10% to 100%. In this range, we find that the rate of convergence for the ratio of quark condensates seems reasonable.

Both F_π and $\langle \bar{u}u\rangle$ show a tendency towards chiral restoration in going from chiral SU(2) to chiral SU(3). This tendency seems much stronger for $\langle \bar{u}u\rangle$. This is in agreement with the arguments of ref. [6] based on the spectrum of the Dirac operator. One must however bear in mind that its dimensionality is different from F_π: if we had considered $\langle \bar{u}u\rangle^{1/3}$, we would have found a smaller variation. This behaviour

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$O(p^6)/O(p^4)$	$10^3\, L_6$	$10^5\,(C_{20}+3C_{21})$	$\langle\bar{u}u\rangle_{\mathrm{SU}(2)}/\langle\bar{u}u\rangle_{\mathrm{SU}(3)}-1$
0.10	0.73 [0.56]	0.29 [−0.02]	$0.85+0.14$
0.30	0.64 [0.47]	0.37 [0.06]	$0.76+0.18$
0.50	0.57 [0.41]	0.43 [0.12]	$0.69+0.21$
1	0.46 [0.30]	0.52 [0.21]	$0.58+0.26$

Table 2: Assuming a given ratio of the $O(p^6)$ to the $O(p^4)$ contributions in the chiral expansion of $\Pi_6(0)$ the values of the LEC's $L_6^r(m_\eta)$ and the combination $C_{20}^r(\mu)+3C_{21}^r(\mu)$ are computed for $\mu = m_\eta$ (in brackets, $\mu = m_\rho$). We also display the successive terms in the expansion of $\langle\bar{u}u\rangle_{\mathrm{SU}(2)}/\langle\bar{u}u\rangle_{\mathrm{SU}(3)}-1$.

could perhaps suggest the possibility, at larger N_F, of a phase with vanishing quark condensate, yet with chiral symmetry still spontaneously broken [25]. Kogan et al. [26] have discussed how this could result from a discrete $\mathbb{Z}_2$ axial subgroup remaining unbroken, but argue against this possibility in QCD. Concerning the error in this evaluation of L_6, finally, it is expected to be somewhat larger than the error on L_4 because of a greater sensitivity to the energy region above 1 GeV. The discussion above also suggests that one should have $L_6(m_\eta) < L_4(m_\eta)$ otherwise the $O(m_s)$ corrections could become out of control. Keeping this mind, we find that L_6 should lie in the following range:

$$L_6(m_\eta) = (0.5 \pm 0.3)\, 10^{-3}\,. \tag{2.31}$$

3. The LEC's in a linear sigma-model

3.1 General scalar meson lagrangian

Here, we adopt a simple resonance saturation point of view for estimating the low-energy coupling constants, which was discussed in detail in ref. [4]. It consists in making a tree level calculation starting from a lagrangian for the resonances. We will consider the scalar resonances here. It is most convenient to start from a representation of the resonances in which they transform under a non-linear representation of the chiral group [27]. A detailed discussion can be found in ref. [4], we adopt essentially the same notations here. If one is interested in the $O(p^4)$ LEC's then one needs consider only those terms which are quadratic in the resonance fields,

$$\mathcal{L}_{SS} = \frac{1}{2}\partial_\mu S_0 \partial^\mu S_0 + \frac{1}{2}\langle\nabla_\mu S \nabla^\mu S\rangle - \frac{1}{2}M_0^2 S_0^2 - \frac{1}{2}M_8^2\langle S^2\rangle\,, \tag{3.1}$$

or containing one resonance field and one scalar source, or one scalar field and two chiral fields u_μ, i.e.

$$\mathcal{L}_{S\chi} = c_d\langle S u_\mu u^\mu\rangle + \tilde{c}_d S_0 \langle u_\mu u_\mu\rangle + c_m\langle S\chi_+\rangle + \tilde{c}_m S_0\langle\chi_+\rangle\,, \tag{3.2}$$

where S_0 is the singlet scalar, with chiral limit mass M_0, S is a traceless matrix encoding the scalar octet, which has a common mass M_8 in the chiral limit. We use the same conventions otherwise as ref. [4]. This lagrangian yields the resonance saturation estimates for the following low-energy constants:

$$\begin{aligned} L_4 &= -\frac{c_d c_m}{3M_8^2} + \frac{\tilde{c}_d \tilde{c}_m}{M_0^2}\,, \qquad & L_5 &= \frac{c_d c_m}{M_8^2}\,, \\ L_6 &= -\frac{c_m^2}{6M_8^2} + \frac{\tilde{c}_m^2}{2M_0^2}\,, \qquad & L_8 &= \frac{c_m^2}{2M_8^2}\,. \end{aligned} \tag{3.3}$$

In addition, the scalar resonances make contributions to L_1, L_3,

$$L_1 = -\frac{c_d^2}{6M_8^2} + \frac{\tilde{c}_d^2}{2M_0^2}\,, \qquad L_3 = \frac{c_d^2}{2M_8^2}\,, \tag{3.4}$$

which receive other important contributions from the vector and axial-vector resonance sector. These tree-level estimates yield coupling constants which are scale independent:[1] one usually assumes that they should represent meaningful estimates of the $L_i^r(\mu)$ for values of μ for which chiral logarithms are numerically small, that is of the order of $\mu = 0.5$ to $\mu = 1\,\mathrm{GeV}$. In ref. [4] the OZI rule was assumed to apply, implying $L_4 = L_6 = 0$. The formulas (3.3) provide one relation between the values of L_5, L_8 and the experimental value of the decay width $a_0 \to \pi\eta$. Values of the couplings c_d and c_m were extracted,

$$|c_d| \simeq 32\,\mathrm{MeV}\,, \qquad |c_m| \simeq 42\,\mathrm{MeV}\,. \tag{3.5}$$

Below, we will investigate the more detailed predictions that one can make if one assumes the validity of a renormalizable, sigma-model type interaction lagrangian. Before we discuss this, let us consider the generalisation of the considerations above to LEC's of chiral order $O(p^6)$. We will illustrate the complications which arise at this order by considering only the lagrangian terms which are cubic in the scalar source χ_+. These terms involve the three coupling constants [10] C_{19}, C_{20}, C_{21},

$$\mathcal{L}^{(6)} = \cdots + C_{19}\langle\chi_+^3\rangle + C_{20}\langle\chi_+^2\rangle\langle\chi_+\rangle + C_{21}\langle\chi_+\rangle^3 + \cdots. \tag{3.6}$$

Firstly, one needs to consider interaction terms which are cubic in the resonance fields,

$$\mathcal{L}_{SSS} = aS_0^3 + bS_0\langle S^2\rangle + c\langle S^3\rangle\,, \tag{3.7}$$

then terms which are quadratic in the resonance fields and linear in the scalar source,

$$\mathcal{L}_{SS\chi} = \mathring{e}_m\ \langle S^2\rangle\langle\chi_+\rangle + \mathring{f}_m\ S_0\langle S\chi_+\rangle + \mathring{g}_m\ \langle S^2\chi_+\rangle + \mathring{h}_m\ S_0^2\langle\chi_+\rangle\,, \tag{3.8}$$

[1] If one starts from the renormalizable sigma-model it is possible, in principle, to compute the one-loop effective action and then generate the $O(p^4)$ LEC's with the correct scale dependence. This was done for the SU(2) sigma-model in ref. [28].

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and, finally, terms linear in the scalar field and quadratic in the scalar source,

$$\mathcal{L}_{S\chi\chi} = a' S_0 \langle \chi_+ \rangle^2 + b' S_0 \langle \chi_+^2 \rangle + c' \langle S\chi_+ \rangle \langle \chi_+ \rangle + d' \langle S \chi_+^2 \rangle \,. \tag{3.9}$$

This already brings in a large number of couplings and it would not be possible to make definite predictions without additional assumptions. The terms in eqs. (3.7), (3.8) and (3.9) generate chiral $O(p^6)$ coupling constants as well as chiral symmetry breaking for the scalar meson masses. It is convenient, both for generating the $O(p^4)$ and $O(p^6)$ LEC's and for expressing the scalar meson masses to make the following field redefinitions:

$$\begin{aligned}
S_0 &\longrightarrow S_0 + \frac{a}{M_0^2} S_0^2 + \frac{b}{M_0^2} \langle S^2 \rangle + \frac{\tilde{c}_m}{M_0^2} \langle \chi_+ \rangle \,, \\
S &\longrightarrow S + \frac{c}{M_8^2} \left(S^2 - \frac{1}{3} \langle S^2 \rangle I_d \right) + \frac{c_m}{M_8^2} \left(\chi_+ - \frac{1}{3} \langle \chi_+ \rangle I_d \right) .
\end{aligned} \tag{3.10}$$

The redefined fields S_0 and S have, by construction, no trilinear couplings and no minimal one-resonance to one-source coupling. The couplings of two resonances to one scalar source, eq. (3.8), get redefined to

$$\begin{aligned}
g_m &= \mathring{g}_m + \frac{3 c_m c}{M_8^2} \,, \qquad & h_m &= \mathring{h}_m + \frac{3 \tilde{c}_m a}{M_0^2} \,, \\
e_m &= \mathring{e}_m + \frac{\tilde{c}_m b}{M_0^2} - \frac{c_m c}{M_8^2} \,, \qquad & f_m &= \mathring{f}_m + \frac{2 c_m b}{M_8^2} \,.
\end{aligned} \tag{3.11}$$

The scalar meson masses, to linear order in the quark masses, can be straightforwardly expressed in terms of these redefined parameters. The masses of the isospin $I = 1$ and $I = 1/2$ mesons read,

$$\begin{aligned}
M_{a_0}^2 &= M_8^2 - 4 e_m \left(2 m_K^2 + m_\pi^2 \right) - 4 g_m m_\pi^2 \,, \\
M_{\kappa_0}^2 &= M_{a_0}^2 - 4 g_m \left(m_K^2 - m_\pi^2 \right) ,
\end{aligned} \tag{3.12}$$

while the isospin zero meson masses diagonalise the singlet-octet mass matrix which gets parametrised as follows:

$$\begin{aligned}
\mathcal{M}_{11} &= M_0^2 - 4 h_m \left(2 m_K^2 + m_\pi^2 \right) , \\
\mathcal{M}_{12} &= 4 f_m \sqrt{\frac{2}{3}} \left(m_K^2 - m_\pi^2 \right) , \\
\mathcal{M}_{22} &= M_{a_0}^2 - \frac{16}{3} g_m \left(m_K^2 - m_\pi^2 \right) .
\end{aligned} \tag{3.13}$$

In deriving these expressions for the masses, we have ignored possible terms in the scalar lagrangian of the form $\partial_\mu S_0 \partial^\mu S_0 \langle \chi_+ \rangle$, $\langle \nabla_\mu S \nabla^\mu S \chi_+ \rangle$, anticipating on the fact that such terms will not appear in the model to be discussed below.

Concerning the LEC's C_{19}, C_{20}, C_{21}, the following expressions are obtained:

$$\begin{aligned}
F_0^{-2} C_{19} &= \frac{c_m^2}{M_8^4}\left(\overset{\circ}{g}_m + \frac{c\, c_m}{M_8^2}\right) + d' \frac{c_m}{M_8^2}\,, \\
F_0^{-2} C_{20} &= \frac{c_m^2}{M_8^4}\left(-\frac{2}{3}\,\overset{\circ}{g}_m + \overset{\circ}{e}_m - \frac{c\, c_m}{M_8^2}\right) + \frac{c_m \tilde{c}_m}{M_0^2 M_8^2}\left(\overset{\circ}{f}_m + \frac{b\, c_m}{M_8^2}\right) + \\
&\quad + \left(c' - \frac{1}{3} d'\right)\frac{c_m}{M_8^2} + b' \frac{\tilde{c}_m}{M_0^2}\,, \\
F_0^{-2}(C_{20} + 3C_{21}) &= -\frac{c_m^2}{3M_8^4}\left(\overset{\circ}{g}_m + \frac{c\, c_m}{M_8^2}\right) + \frac{3\tilde{c}_m^2}{M_0^4}\left(\overset{\circ}{h}_m + \frac{a\, \tilde{c}_m}{M_0^2}\right) + \\
&\quad + (3a' + b')\frac{\tilde{c}_m}{M_0^2} - d' \frac{c_m}{3M_8^2}\,. \qquad (3.14)
\end{aligned}$$

3.2 Parameters of the linear sigma-model

In the SU(3) linear sigma-model, one first encodes the scalar nonet and the pseudo-scalar nonet into a complex 3×3 matrix Σ, which, under a chiral transformation transforms as,

$$\Sigma \longrightarrow g_R \Sigma g_L^\dagger\,, \qquad (3.15)$$

and the lagrangian is assumed to be renormalizable. In the chiral limit, the most general renormalizable lagrangian, invariant under the chiral group (except for the $U_A(1)$ subgroup) contains four parameters,

$$\begin{aligned}
\mathcal{L} = \frac{1}{2}\left\langle \partial_\mu \Sigma\, \partial^\mu \Sigma^\dagger \right\rangle - \frac{1}{2}\mu^2 \left\langle \Sigma\Sigma^\dagger \right\rangle - \lambda \left\langle \left(\Sigma\Sigma^\dagger\right)^2 \right\rangle - \\
- \lambda' \left\langle \Sigma\Sigma^\dagger \right\rangle^2 + \beta\left(\det(\Sigma) + \det\left(\Sigma^\dagger\right)\right). \qquad (3.16)
\end{aligned}$$

This lagrangian was reconsidered recently by several authors [15, 29, 30, 31] and we use the same notations as in ref. [31]. Suitable choices of the parameters $\mu^2, \lambda, \lambda', \beta$ ensure that spontaneous chiral symmetry breaking occurs, i.e. that the potential is minimised for a non-zero value of the vacuum expectation value of the Σ matrix,

$$\bar{\Sigma} = v I_d\,, \qquad (3.17)$$

(which we take to be diagonal, assuming that no spontaneous breaking of flavour symmetry occurs). Values of v which correspond to extremums of the potential are solutions of the cubic equation,

$$v\left((4\lambda + 12\lambda')v^2 - 2\beta v + \mu^2\right) = 0\,, \qquad (3.18)$$

which has real solutions $v \neq 0$ provided the parameters satisfy

$$\beta^2 \geq \mu^2 (4\lambda + 12\lambda')\,. \qquad (3.19)$$

Next, expanding around the minimum we find that the pseudoscalars are massless, except for the singlet, η_0 which mass is proportional to β,

$$M^2_{\eta_0} = 6\beta v\,. \tag{3.20}$$

We can also read off the expressions for the singlet and octet scalar meson masses in the chiral limit, M_0 and M_8,

$$M_8^2 = 8\lambda v^2 + 4\beta v\,, \qquad M_0^2 = 8(\lambda + 3\lambda')v^2 - 2\beta v\,. \tag{3.21}$$

Stability of the vacuum requires that these squared masses be positive. Using eqs. (3.18), (3.20) and (3.21) we can trade the original four parameters of the lagrangian (3.16) for the more physically relevant ones v, M_{η_0}, M_0 and M_8.

In QCD, chiral symmetry is broken by the light quark masses. We will make the simplifying assumption, at the level of the sigma-model, to consider symmetry breaking to linear order in the quark masses but we accept all the terms which are soft (i.e. of dimensionality strictly smaller than four). Under these assumptions the most general symmetry breaking sector has, again, four parameters,

$$\begin{aligned}\mathcal{L}_{SB} = {} & \gamma_0 \left\langle \Sigma\chi^\dagger \right\rangle + \gamma_1 \left\langle \Sigma^{-1}\chi \right\rangle \det(\Sigma) + \gamma_2 \left\langle \Sigma\Sigma^\dagger\Sigma\chi^\dagger \right\rangle + \\ & + \gamma_3 \left\langle \Sigma\Sigma^\dagger \right\rangle \left\langle \Sigma\chi^\dagger \right\rangle + \text{h.c.}\,.\end{aligned} \tag{3.22}$$

Usually, only the first term is considered. This increased phenomenological flexibility will allow us to obtain better fits of the scalar nonet masses. While more general, this lagrangian will prove nevertheless to be reasonably constraining. We note that only the first two terms are renormalizable in the strict sense that the counterterms are exactly of the same form [17]. The last two terms generate counterterms which are of higher order in the scalar source χ. Consistent with the assumption of not including further terms quadratic in the quark mass matrix at this level, we will compute the scalar meson masses and the vacuum expectation values at linear order as well. We have checked this approximation by comparing with exactly calculated masses.

The connection with the representation of the scalar fields used in section 3.1 is performed by making a change of variables [32]

$$\Sigma = \exp\left(i\sqrt{\frac{2}{3}}\frac{\eta_0}{F_0}\right) uHu \tag{3.23}$$

with

$$u = \exp\left(i\sum_1^8 \frac{\lambda_a\pi_a}{2F_0}\right), \qquad H = \left(v + \frac{S_0}{\sqrt{3}}\right) I_d + S\,. \tag{3.24}$$

This change of variable makes sense only if $F_0 \neq 0$, i.e. when chiral symmetry is spontaneously broken. The kinetic energy part of the linear sigma-model lagrangian gets transformed into

$$\mathcal{L}_{kin} = \frac{1}{2}\partial_\mu S_0 \partial^\mu S_0 + \frac{1}{2}\langle \nabla_\mu S \nabla^\mu S\rangle + \frac{1}{8}\left\langle \{u_\mu, H\}^2\right\rangle + \frac{1}{3F_0^2}\partial_\mu \eta_0 \partial^\mu \eta_0 \langle H^2\rangle - \\ - \sqrt{\frac{2}{3}}\frac{\partial_\mu \eta_0}{F_0}\langle H^2 u^\mu\rangle \tag{3.25}$$

with

$$\nabla_\mu S = \partial_\mu S + [\Gamma_\mu, S]\,, \quad \Gamma_\mu = \frac{1}{2}\left(u^\dagger \partial_\mu u + u\partial_\mu u^\dagger\right), \quad u_\mu = i\left(u^\dagger \partial_\mu u - u\partial_\mu u^\dagger\right). \tag{3.26}$$

Replacing H by its expression above (3.24) one identifies v with the pion decay constant in the chiral limit F_0 and one obtains a prediction for the two couplings c_d and $\tilde{c}_d$,

$$v^2 = \frac{F_0^2}{2}\,, \qquad c_d = v\,, \qquad \tilde{c}_d = \frac{v}{\sqrt{3}}\,. \tag{3.27}$$

The symmetry breaking lagrangian, expressed in terms of the new fields reads

$$\mathcal{L}_{SB} = \left\langle \left(\gamma_0 H + \gamma_2 H^3 + \gamma_3 \langle H\rangle^2 H\right)\left(\chi_+ \cos\sqrt{\frac{2}{3}}\frac{\eta_0}{F_0} - i\chi_- \sin\sqrt{\frac{2}{3}}\frac{\eta_0}{F_0}\right)\right\rangle + \\ + \gamma_1 \det H \left\langle H^{-1}\left(\chi_+ \cos\sqrt{\frac{8}{3}}\frac{\eta_0}{F_0} + i\chi_- \sin\sqrt{\frac{8}{3}}\frac{\eta_0}{F_0}\right)\right\rangle. \tag{3.28}$$

In this sector, we obtain one relation among the four parameters γ_i from the requirement that the coefficient of the chiral lagrangian term $\langle \chi_+\rangle$ be correctly normalized to $F_0^2/4$ and we can express two other parameters in terms of the meson-source couplings c_m and $\tilde{c}_m$. This gives the relations

$$\begin{aligned} \gamma_0 &= \frac{3}{4}v + \frac{1}{6}c_m - \frac{2}{\sqrt{3}}\tilde{c}_m + \gamma_3 v^2\,, \\ \gamma_1 v &= -\frac{1}{3}\left(c_m - \sqrt{3}\tilde{c}_m\right) - 2\gamma_3 v^2\,, \\ \gamma_2 v^2 &= -\frac{1}{4}v + \frac{1}{6}c_m + \frac{1}{\sqrt{3}}\tilde{c}_m - 2\gamma_3 v^2\,. \end{aligned} \tag{3.29}$$

In this sector, we can take c_m, $\tilde{c}_m$ and γ_3 as arbitrary parameters.

In this model, we can now express the many parameters that appeared in the general scalar lagrangian discussed above. Firstly, for the terms coupling one scalar field to two scalar sources we have simply $a' = b' = c' = d' = 0$. For the trilinear couplings of resonances, one obtains the relations

$$\begin{aligned} a &= -\frac{1}{2\sqrt{3}v}\left(M_0^2 + \frac{1}{9}M_{\eta_0}^2\right), \\ b &= -\frac{1}{2\sqrt{3}v}\left(M_0^2 + 2M_8^2 - \frac{2}{3}M_{\eta_0}^2\right), \\ c &= -\frac{1}{2v}\left(M_8^2 - \frac{8}{9}M_{\eta_0}^2\right). \end{aligned} \tag{3.30}$$

Finally, the parameters g_m, h_m, e_m, f_m which control linear symmetry breaking of the scalar meson masses (see eqs. (3.12) and (3.13)) obey the following relations:

$$
\begin{aligned}
g_m &= \frac{4}{3v}\left(\sqrt{3}\tilde{c}_m - c_m + c_m \frac{M_{\eta_0}^2}{M_8^2}\right) - \frac{3}{4} - 8\gamma_3 v\,, \\
\sqrt{3} f_m &= \frac{1}{v}\left(\sqrt{3}\tilde{c}_m - c_m \frac{M_0^2}{M_8^2}\right) - \frac{9}{8} + \frac{1}{2} g_m\,, \\
e_m &= \frac{M_{\eta_0}^2 - 3M_8^2}{9M_0^2 v}\left(\sqrt{3}\tilde{c}_m - c_m \frac{M_0^2}{M_8^2}\right) - \frac{3}{16} - \frac{1}{4} g_m\,, \\
h_m &= -\frac{M_{\eta_0}^2}{18M_0^2 v}\left(\sqrt{3}\tilde{c}_m - c_m \frac{M_0^2}{M_8^2}\right) - \frac{9}{32} - \frac{1}{24} g_m\,.
\end{aligned}
\tag{3.31}
$$

Making use of the relations (3.11) we can express the three $O(p^6)$ LEC's C_{19}, C_{20}, C_{21} discussed above in terms of the parameters of the linear sigma-model. We are also interested in the prediction for the OZI suppressed $O(p^6)$ LEC C_{16} which participates in the flavour variation of the order parameter F_π as discussed above. After a small calculation, the following expression is obtained,

$$
\begin{aligned}
F_0^{-2} C_{16} = &\frac{1}{54}\left(2c_m + 10\sqrt{3}\tilde{c}_m - 9v\right)\left(\frac{1}{M_0^2} - \frac{1}{M_8^2}\right)\left(\frac{2c_m}{M_8^2} + \frac{\sqrt{3}\tilde{c}_m}{M_0^2}\right) + \\
&+ \frac{\gamma_3 v^2}{9}\left(\frac{2\sqrt{3}\tilde{c}_m}{M_0^2}\left(\frac{1}{M_0^2} + \frac{2}{M_8^2}\right) - \frac{4c_m}{M_8^2}\left(\frac{2}{M_0^2} - \frac{5}{M_8^2}\right)\right) + \\
&+ \frac{M_{\eta_0}^2}{54M_0^2 M_8^2}\left(\frac{2c_m}{M_8^2} + \frac{\sqrt{3}\tilde{c}_m}{M_0^2}\right)\left(2c_m\left(1 - 2\frac{M_0^2}{M_8^2}\right) - \sqrt{3}c_m \frac{M_8^2}{M_0^2}\right).
\end{aligned}
\tag{3.32}
$$

3.3 Phenomenological applications

Altogether, the sigma-model as considered here has six independent parameters (apart from F_0): three chiral limit masses M_0, M_8 and M_{η_0} and three couplings to the scalar source c_m, $\tilde{c}_m$ and γ_3. The parameter M_{η_0}, i.e. the mass of the η' in the chiral limit can be estimated to be [4],

$$
M_{\eta_0} \simeq 0.8\,\mathrm{GeV}\,. \tag{3.33}
$$

Let us now discuss the determination of the remaining five parameters. A priori, one expects to be able to reproduce exactly the four independent masses (M_{a_0}, M_{κ_0}, M_{σ_0}, M_{f_0}) in the scalar nonet and, as an additional constraint a natural choice would be to enforce the super-convergence relation (2.15) which implies here $c_m = \sqrt{3}\tilde{c}_m$. In practice, however, because of non-linearities, this choice strongly restricts the possible range of masses for the κ_0. This is an indication that this model should be considered as a toy model rather than a really physically meaningful one. Instead, we did not try to impose the super-convergence relation and use a constraint from the pseudo-scalar sector, the ratio F_K/F_π which allows for a fairly broad range of values for M_{κ_0}. Because of the specific parametrisation of the mass matrix, eq. (3.13) its entries get evaluated as follows in terms of physical masses,

$$\mathcal{M}_{22} = \frac{4M_{\kappa_0}^2 - M_{a_0}^2}{3}\,, \qquad \mathcal{M}_{11} = M_{\sigma_0}^2 + M_{f_0}^2 - \mathcal{M}_{22}\,, \tag{3.34}$$

and, using the determinant,

$$\mathcal{M}_{12} = \pm\sqrt{-\left(\mathcal{M}_{22} - M_{\sigma_0}^2\right)\left(\mathcal{M}_{22} - M_{f_0}^2\right)}\,. \tag{3.35}$$

We observe that there are two possible sign assignments. Reality of $\mathcal{M}_{12}$ requires that the following inequality be satisfied

$$\frac{M_{a_0}^2 + 3M_{\sigma_0}^2}{4} \le M_{\kappa_0}^2 \le \frac{M_{a_0}^2 + 3M_{f_0}^2}{4}\,. \tag{3.36}$$

As shown in [33], this inequality holds in a model independent way if symmetry breaking is assumed to be linear. From the expression of the mass matrix (3.13) it is then easy to see that the scalar singlet chiral mass M_0^2 satisfies a quadratic equation

$$M_0^4 - BM_0^2 + CM_{\eta_0}^2 = 0\,, \tag{3.37}$$

with

$$\begin{aligned} B &= \mathcal{M}_{11} - \frac{9}{8}\left(2m_K^2 + m_\pi^2\right)\left(1 - \frac{1}{27}\frac{M_{\kappa_0}^2 - M_{a_0}^2}{m_K^2 - m_\pi^2}\right), \\ C &= \frac{1}{4}\left(2m_K^2 + m_\pi^2\right)\left(1 + \frac{1}{9}\frac{M_{\kappa_0}^2 - M_{a_0}^2}{m_K^2 - m_\pi^2} + \frac{\sqrt{2}}{3}\frac{\mathcal{M}_{12}}{m_K^2 - m_\pi^2}\right). \end{aligned} \tag{3.38}$$

Real solutions exist provided

$$B^2 - 4CM_{\eta_0}^2 \ge 0\,, \tag{3.39}$$

and it must further be checked that at least one solution is positive. This puts further constraints on the allowed values of the scalar meson masses in this model. Once M_0^2 is determined, the other parameters are easily evaluated. For instance M_8^2 is given by,

$$\begin{aligned} M_8^2 = \frac{1}{1 + \frac{6C}{M_0^2}}\Bigg[&M_{a_0}^2 + \frac{2CM_{\eta_0}^2}{M_0^2} - \frac{3}{4}\left(2m_K^2 + m_\pi^2\right) + \\ &+ \frac{1}{4}\left(2m_K^2 - 3m_\pi^2\right)\frac{M_{\kappa_0}^2 - M_{a_0}^2}{m_K^2 - m_\pi^2}\Bigg]. \end{aligned} \tag{3.40}$$

Next, using as experimental input $M_{\kappa_0}^2 - M_{a_0}^2$ and $\mathcal{M}_{12}$ together with eqs. (3.12), (3.13) and (3.31)) gives two linear equations for the three quantities c_m/v, $\tilde{c}_m/v$ and $\gamma_3 v$.

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We will use as an additional constraint, the ratio F_K/F_π, which determines c_m/v from the relation,

$$\frac{c_m}{v} = \frac{M_8^2}{2F_\pi}\frac{F_K - F_\pi}{m_K^2 - m_\pi^2}\,. \tag{3.41}$$

Finally, there remains to determine $v = F_0/\sqrt{2}$. To linear order in the quark masses, v is given by the following expression:

$$v = \frac{F_\pi}{\sqrt{2}}\left[1 - \frac{2}{3}m_\pi^2\left(\frac{2c_m/v}{M_8^2} + \frac{\sqrt{3}\tilde{c}_m/v}{M_0^2}\right) - \frac{4}{3}m_K^2\left(\frac{-c_m/v}{M_8^2} + \frac{\sqrt{3}\tilde{c}_m/v}{M_0^2}\right)\right]. \tag{3.42}$$

This expression can be recovered in two different ways: one can either use CHPT together with the expressions (3.3) for L_4 and L_5 or write down the equations for the vacuum expectation values in the presence of quark masses and solve these equations to linear order in the quark masses.

3.3.1 Light σ meson

The existence of a very broad scalar resonance in $\pi\pi$ scattering with $M \simeq \Gamma \simeq 0.5 - 0.6\,\mathrm{GeV}$ is by now well established since the work of Basdevant Froggatt and Petersen [18], in which the whole set of constraints from unitarity, analyticity, and crossing symmetry has been implemented (see ref. [35] for a recent survey and the particle data book [5] for a complete list of references). What is unclear is whether this state should be interpreted as a physical light scalar resonance. Recently , Black et al. [20] have proposed arguments based on perturbative unitarity favouring the existence of a light σ and also, of a light κ meson. According to the inequality (3.36), if the sigma meson is light then one must have $M_{\kappa_0} \leq M_{a_0}$ unless our assumption of linearity in the quark masses fails. Recently, Törnqvist [31] (similar fits were also discussed in ref. [15]) has attempted to accommodate in the linear sigma-model a scalar meson multiplet with a light sigma and a heavy κ. Such fits fail to obtain the $f_0(980)$ at the correct mass and therefore would not correctly reproduce the $\pi\pi$ phase-shifts around 1 GeV, even if one-loop corrections are included [34].

Let us now assume that a light sigma meson exists, e.g. $M_{\sigma_0} \simeq 0.6\,\mathrm{GeV}$, and discuss the consequences. We take otherwise $M_{a_0} = 0.983\,\mathrm{GeV}$, and $M_{f_0} = 0.980\,\mathrm{GeV}$ from experiment and $M_{\kappa_0} = 0.9\,\mathrm{GeV}$ as proposed in ref. [20]. From this input one easily calculates the parameters A and B in the quadratic equation (3.37),

$$A = 0.024\,\mathrm{GeV}^2\,, \qquad B = 0.042\,\mathrm{GeV}^2\,. \tag{3.43}$$

There results that no real solution to the equation for M_0^2 exists unless $M_{\eta_0} \leq 0.06\,\mathrm{GeV}$, which appears as an absurdly small value. Reality and positivity of M_0^2 (and M_8^2) are necessary conditions for the existence of a stable minimum of the potential, with spontaneous chiral symmetry breaking. One must thus conclude that with such input scalar meson masses $N_F = 3$ chiral symmetry is not spontaneously broken in the linear sigma-model. The same conclusion holds if one picks up larger

M_{κ_0}	M_0	M_8	c_m	$\sqrt{3}\,\tilde{c}_m$	γ_3	$\sqrt{2}\,v$
1.10	1.17	0.85	0.023	0.040	0.54	0.092
1.20	1.05	0.93	0.028	0.027	0.33	0.094
1.30	0.87	0.90	0.023	0.034	1.56	0.081
1.33	0.79	0.85	0.017	0.037	2.79	0.068
1.35	0.72	0.80	0.011	0.033	4.82	0.051

Table 3: Determination of parameters in the case of a heavy scalar nonet $M_{\sigma_0} = 0.980$, $M_{f_0} = 1.5$ for several values of M_{κ_0}. All entries are in GeV except γ_3 which is in GeV^{-1}.

values for M_{κ_0}: using the set of parameters determined in ref. [15] for $M_\sigma = 0.4$ and $M_\sigma = 0.6\,\mathrm{GeV}$ one verifies that the equation for the vacuum expectation value v eq. (3.18) has no real solution $v \neq 0$. A better situation prevails if we accept smaller values for M_{κ_0}. If we take, for instance, $M_{\kappa_0} = 0.750\,\mathrm{GeV}$, which is the smallest value allowed by the inequality (3.36), then, eq. (3.37) has real solutions provided $M_{\eta_0} \leq 0.62\,\mathrm{GeV}$, which is still reasonable. A problem remains, however, that being close to a situation where chiral symmetry is unbroken for $N_F = 3$, while it is in general broken for $N_F = 2$, the expansion in the strange quark mass is likely to be non converging. Indeed, if we try to compute F_0 with, say, $M_{\eta_0} = 0.50\,\mathrm{GeV}$, using our linear expansion formula (3.42), we find $F_0 \simeq 28\,\mathrm{MeV}$, which is very small compared to F_π, suggesting that the expansion in m_s is not perturbative in this case.

3.3.2 Heavy σ meson

Let us now consider the scenario of a heavy sigma meson, i.e. we identify the sigma meson with the well established resonance $f_0(980)$ and we identify the heavier $I = 0$ member of the nonet with the $f_0(1500)$ (see [24]). With these assignments, we find that a solution of eq. (3.37) with real M_0^2 exists in the range

$$M_{\kappa_0} \leq 1.37\,\mathrm{GeV}\,, \tag{3.44}$$

taking $M_{\eta_0} = 0.8\,\mathrm{GeV}$ and the negative sign for $\mathcal{M}_{12}$ (see (3.35)). This means that we cannot exactly reproduce the experimental mass of the $K_0^*(1430)$ but we can get reasonably close. We will eventually allow M_{κ_0} to vary somewhat away from the experimental result. Numerical values of the parameters M_0, M_8, c_m, $\tilde{c}_m$, γ_3 as well as v determined for several values of M_{κ_0} are collected in table 3. We note that the situation with a minimal symmetry breaking lagrangian (i.e. $\gamma_1 = \gamma_2 = \gamma_3 = 0$) is close to the case when $M_{\kappa_0} = 1.20$ in the table. In this case, one has $c_m = \sqrt{3}\tilde{c}_m = v/2$. The value of $F_0 = \sqrt{2}v$ remains reasonable close to F_π except when M_{κ_0} gets very near to its upper bound. The coupling c_m comes out smaller than that determined in ref. [4] (see (3.5)) while the coupling $c_d = v$ comes out larger. The coupling c_d controls the decay width $a_0 \to \pi\eta$, which tends to be too large in this model (accepting that the experimental value is $\Gamma \simeq 60\,\mathrm{MeV}$, which is subject to some debate [5]), the result in this respect improves as M_{κ_0} gets larger.

M_{κ_0}	$10^3\, L_5$	$10^3\, L_7$	$10^3\, L_8$	$10^5\, C_{19}$	$10^5\, C_{20}$
1.10	2.06	-0.22	0.36	-0.17	-0.05
1.20	2.15	-0.47	0.45	-0.34	0.07
1.30	1.61	-0.62	0.32	-0.45	0.12
1.33	1.13	-0.51	0.20	-0.38	0.08
1.35	0.64	-0.31	0.10	-0.24	0.04

Table 4: Numerical values of some $O(p^4)$ and $O(p^6)$ LEC's from the sigma-model with a heavy sigma.

Once these couplings are known it is easy to determine the LEC's with the formulas given above. Numerical values for the LEC's dominated by the scalar mesons and not suppressed in the large-N_c limit are shown in table 4. We have also included L_7, which is assumed to be saturated by the η_0, computed from the expression [4],

$$L_7 = -\frac{\tilde{d}_m^2}{2M_{\eta_0}^2}\,, \tag{3.45}$$

where the coupling $\tilde{d}_m$ of the η_0 field to the pseudo-scalar current has the following expression in the present version of the linear sigma-model,

$$\tilde{d}_m = -\frac{F_0}{2\sqrt{6}}\left(1+\frac{2}{v}\left(c_m-\sqrt{3}\tilde{c}_m\right)+12\gamma_3 v\right). \tag{3.46}$$

The fact that L_5 does not remain constant when M_{κ_0} increases is a reflection of the effect of non-linearities in m_s. The values of the LEC's may be compared with the ones determined at $O(p^4)$ from experimental data [1]

$$\begin{aligned} &10^3\, L_5^r(m_\eta) = 2.2\pm 0.5\,, \qquad 10^3\, L_7 = -0.4\pm 0.15\,,\\ &10^3\, L_8^r(m_\eta) = 1.1\pm 0.3\,. \end{aligned} \tag{3.47}$$

While the prediction for L_7 is acceptable, that for L_8 is too small. This is related to the small size of the meson to source coupling c_m predicted by the model. Still, the order of magnitude is correct except, perhaps, for the very last line in the table. The LEC's were recently re-determined in an $O(p^6)$ analysis of the experimental data [36] and this brings the values somewhat down in magnitude: $10^3\, L_5^r(m_\eta) = 1.45\pm 0.12$, $10^3\, L_7 = -0.31\pm 0.15$, $10^3\, L_8^r(m_\eta) = 0.68\pm 0.18$. A resonance-saturated estimate for the $O(p^6)$ coupling C_{19} was provided in ref. [12]. The value that we obtain is significantly smaller, by a factor $3-4$. One difference in the estimates is that we include the effect of cubic interaction terms, but the numerical influence of these terms turns out to be unimportant. The main reason for our smaller result is, again, the fact that the coupling c_m is smaller here.

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M_{κ_0}	$10^3\,L_4$	$10^3\,L_6$	$10^5\,C_{16}$	$10^5\,(C_{20}+3C_{21})$
1.10	-0.06	0.07	0.19	0.02
1.20	-0.16	-0.04	0.08	0.05
1.30	0.33	0.15	0.21	0.08
1.33	0.56	0.28	0.36	0.15
1.35	0.55	0.31	0.41	0.27

Table 5: Numerical values of OZI suppressed $O(p^4)$ and $O(p^6)$ LEC's from the sigma-model with a heavy sigma.

Let us now return to the OZI suppressed coupling constants L_4 and L_6 and their $O(p^6)$ counterparts C_{16} and $C_{20}+3C_{21}$. The predictions from the linear sigma-model are collected in table 5. The values and signs of L_4, L_6 are seen to depend very much on the mass of κ_0. They both approximately vanish when $M_{\kappa_0} \simeq 1.24\,\mathrm{GeV}$. This is the point of minimal OZI violation. As one pushes the κ_0 to higher masses, as required by experiment, then the pattern of OZI violation is not unlike the one found on the basis of sum rules. Both L_4 and L_6 become positive, and the orders of magnitudes seem to be in agreement with the discussion based on sum rules. There is even a relatively good agreement as far as L_4 and C_{16} are concerned.

4. Conclusions

In this paper, we have pursued an investigation on the evolution of order parameters of the QCD chiral vacuum as one increases the number of massless flavours, as can be inferred from experimental data. An information on this evolution is contained in the $O(p^4)$ coupling constants L_4 and L_6 which control how F_π and $\langle \bar{u}u \rangle$, respectively vary from the SU(2) chiral limit with $m_s \simeq 200\,\mathrm{MeV}$ to the SU(3) chiral limit. Estimates for the couplings L_4 and L_6 are based on the scalar form factors of the pion and the Kaon which are reconstructed from experimental S-wave scattering data modulo some assumptions. We have investigated the $O(m_s)$ corrections to our previous results. These have two origins:

(a) the normalization of the form factors at the origin must include the $O(p^4)$ contributions.

(b) the chiral expansions (for F_π and $\Pi_6(0)$) must be pursued up to order p^6.

These corrections seem not to affect in a significant way the results obtained previously. In particular there is no sign that these corrections go in the sense of decreasing the values of L_4 and L_6 to make them compatible with the naive large-N_c expectation. This conclusion, however, only holds provided one makes the assumption that the chiral expansion is reasonable, in other terms, that the $O(p^6)$ contribution (say in the expansion of $\Pi_6(0)$) is not larger than the $O(p^4)$ contribution. Modulo this assumption, one also obtains from this analysis an estimate of the $O(p^6)$

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coupling constant C_{16} and the combination $C_{20} + 3C_{21}$. Influence of the values of L_4, L_6 on the convergence rate of the chiral expansion for various observables was studied very recently in [37]. Because of such assumptions which enter into the calculation, the error bars on our coupling constant estimates must be considered as educated guesses. Our results are compatible with a tendency towards chiral restoration upon increasing N_F from $N_F = 2$ to $N_F = 3$ which is significant, particularly so for the quark condensate (in agreement with the arguments of [6]), for which the decrease by a factor of approximately two found in ref. [7] is confirmed. This behaviour of the quark condensate is in qualitative agreement with the one obtained from the instanton liquid model [38]. It would be also interesting to compare this result with unquenched lattice simulations with $N_F = 3$. Unfortunately, such simulations are not available yet, but will exist in the near future.

We also discussed some predictions from the linear sigma-model, considered as a simple toy model. One application is to gauge the possible importance of self-couplings of resonances (which are present in the model) in resonance saturation estimates of the $O(p^6)$ LEC's. Furthermore, the model provides a connection between simply the spectrum of the scalars and the evolution of chiral order parameters with N_F. In order to improve the ability of the model to reproduce experimental masses, the symmetry breaking sector was made more general than usually done, but still linear in the quark mass matrix. We have considered two possibilities for the light scalar nonet:

(*a*) The nonet contains a light σ and a light κ. In this case, we find that chiral symmetry tends to be restored already at $N_F = 3$ or, at least, for a value of N_F somewhat too small, making the chiral expansion in m_s unreliable.

(*b*) The nonet is composed of the resonances $a_0(980)$, $f_0(980)$, $K_0^*(1430)$ and $f_0(1500)$ (which are well established). In this case, we find that we need only a small amount of nonlinear effects to exactly reproduce the $K_0^*(1430)$ mass, and SU(3) chiral symmetry is realised in the Goldstone mode. Concerning L_4 and L_6, we find that they become different from zero as one increases the mass of the $K_0^*(1430)$ towards its experimental value. The signs and orders of magnitude, then, are compatible with the preceding analysis. The same is true of the related $O(p^6)$ constants.

Acknowledgments

The author thanks B. Ananthanarayan for drawing his attention to ref. [24] and Jan Stern for useful remarks. This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative research agreement #DF-FC02-94ER40818 and by the EURODAPHNE network EEC contract #TMR-ERBFMRX-CT980169.

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A. The correlation function $\Pi_6(s)$ to $O(p^6)$

At chiral order p^4, a simple calculation gives the correlation function $\Pi_6(s)$,

$$\Pi_6(s) = 2\bar{J}_K(s) + \frac{4}{9}\bar{J}_\eta(s) + 64L_6^r - \frac{1}{16\pi^2}\left[2L_K + \frac{4}{9}L_\eta + \frac{22}{9}\right], \tag{A.1}$$

where $\bar{J}_P(s)$ is the one-loop function defined as to vanish for $s = 0$,

$$\bar{J}_P(s) = \frac{1}{16\pi^2}\left(\sigma_P \log\frac{\sigma_P - 1}{\sigma_P + 1} + 2\right), \qquad \sigma_P = \sqrt{1 - \frac{4M_P^2}{s}}. \tag{A.2}$$

In all the formulas M_P^2 stands not for the physical pseudo-scalar meson masses but for their lowest order chiral expansion,

$$M_\pi^2 \equiv 2mB_0\,, \qquad M_K^2 \equiv (m + m_s)B_0\,, \qquad M_\eta^2 \equiv \frac{(2m + 4m_s)B_0}{3}. \tag{A.3}$$

The following simplified notation for logarithms was introduced

$$L_P = \log\frac{M_P^2}{\mu^2}. \tag{A.4}$$

At order p^6, one must compute the one-loop and two-loops diagrams shown in figure 1 and add the tree-level contributions from the $O(p^6)$ chiral lagrangian. The calculation

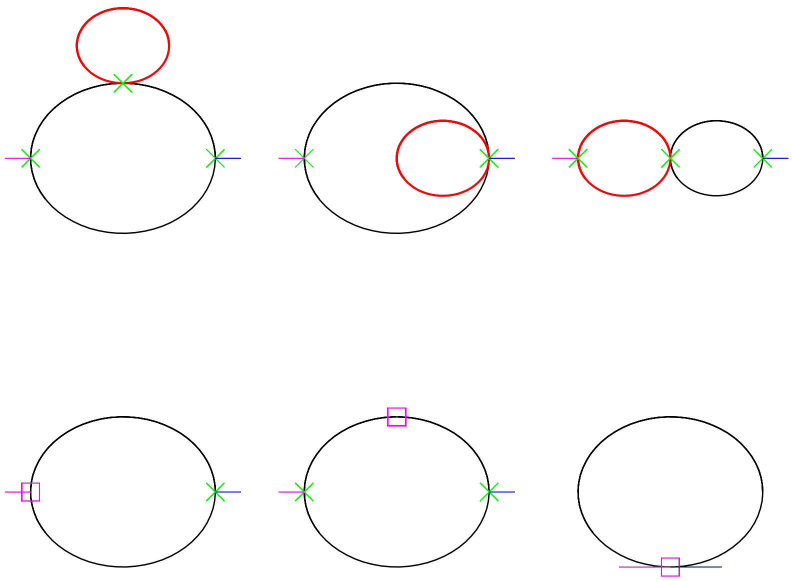

Figure 1: Set of one- and two-loops Feynman graphs contributing at chiral order p^6 to the correlation function $\Pi_6(s)$: crosses denote vertices from the $O(p^2)$ chiral lagrangian, boxes denote vertices from the $O(p^4)$ chiral lagrangian. Some diagrams obtained by exchanging external lines have not been drawn.

is lengthy but not excessively difficult because of the absence of sunset-type diagrams in the present case. The result may be written as a sum of six terms:

$$\Delta\Pi_6(s) = \Delta\Pi_6^a(s) + \Delta\Pi_6^b(s) + \Delta\Pi_6^{\rm resc}(s) + A\frac{s}{F_0^2} + B + C\,. \tag{A.5}$$

The first term encodes the $O(p^4)$ corrections to the pseudo-scalar meson masses in the one-loop functions,

$$\Delta\Pi_6^a(s) = -s\left(2\,\frac{\Delta M_K^2}{M_K^2}\frac{d\bar{J}_K(s)}{ds} + \frac{4}{9}\,\frac{\Delta M_\eta^2}{M_\eta^2}\frac{d\bar{J}_\eta(s)}{ds}\right). \tag{A.6}$$

These mass corrections are given by the following expressions [1]:

$$\begin{aligned}\Delta M_K^2 = M_K^2\Bigg[&\frac{mB_0}{F_0^2}\left(-32S_6 - 8S_8 + \frac{1}{72\pi^2}L_\eta\right) + \\ &+ \frac{m_sB_0}{F_0^2}\left(-16S_6 - 8S_8 + \frac{1}{36\pi^2}L_\eta\right)\Bigg]\end{aligned} \tag{A.7}$$

and

$$\begin{aligned}\Delta M_\eta^2 = M_\eta^2\Bigg[&\frac{mB_0}{F_0^2}\left(-32S_6 - \frac{16}{3}S_8 - \frac{5}{48\pi^2}L_\pi + \frac{1}{9\pi^2}L_K - \frac{1}{144\pi^2}L_\eta\right) + \\ &+ \frac{m_sB_0}{F_0^2}\left(-16S_6 - \frac{32}{3}S_8 + \frac{1}{24\pi^2}L_\pi + \frac{1}{18\pi^2}L_K - \frac{1}{18\pi^2}L_\eta\right)\Bigg] + \\ &+ \frac{(m_s-m)^2B_0^2}{F_0^2}\left[\frac{128}{9}S_7 + \frac{1}{108\pi^2}L_K - \frac{1}{18\pi^2}L_\pi\right].\end{aligned} \tag{A.8}$$

In these expressions, S_6, S_7 and S_8 denote the following combinations of $O(p^4)$ coupling constants:

$$S_6 = -2L_6^r + L_4^r\,, \qquad S_7 = 3L_7^r + L_8^r\,, \qquad S_8 = -2L_8^r + L_5^r\,. \tag{A.9}$$

The second corrective term $\Delta\Pi_6^b(s)$ is generated by $O(p^4)$ corrections to the scalar form factors and has the following form:

$$\begin{aligned}\Delta\Pi_6^b(s) = 3\bar{J}_\pi(s)&\left[\frac{d\Delta M_\pi^2}{dm_sB_0} + \frac{s}{F_0^2}\left(8L_4^r - \frac{1}{32\pi^2}(L_K+1)\right)\right] + \\ + \bar{J}_K(s)&\Bigg[2\,\frac{d\Delta M_K^2}{dmB_0} + 2\,\frac{d\Delta M_K^2}{dm_sB_0} + \\ &+ \frac{s}{F_0^2}\left(16L_5^r + 48L_4^r - \frac{1}{32\pi^2}(3L_\pi + 6L_K + 3L_\eta + 12)\right)\Bigg] + \\ + \frac{1}{3}\bar{J}_\eta(s)&\left[2\frac{d\Delta M_\eta^2}{dmB_0} + \frac{d\Delta M_\eta^2}{dm_sB_0} + \frac{s}{F_0^2}\left(\frac{32}{3}L_5^r + 40L_4^r - \frac{9}{32\pi^2}(L_K+1)\right)\right].\end{aligned} \tag{A.10}$$

The derivatives of ΔM_K^2 and ΔM_η^2 are easily computed from the formulas above and for ΔM_π^2 one has,

$$\frac{d\Delta M_\pi^2}{m_s B_0} = mB_0 \left(-32 S_6 - \frac{1}{36\pi^2}(L_\eta + 1) \right). \tag{A.11}$$

The third corrective term $\Delta\Pi_6^{\rm resc}(s)$ is the rescattering contribution,

$$\begin{aligned} \Delta\Pi_6^{\rm resc}(s) = & \frac{3s}{2F_0^2} \bar{J}_K(s) \left(\bar{J}_\pi(s) + \bar{J}_K(s) + \bar{J}_\eta(s) \right) + \\ & + \frac{4}{3}\bar{J}_\eta(s) \left(\frac{(m+8m_s)B_0}{27F_0^2}\bar{J}_\eta(s) - \frac{(m+m_s)B_0}{F_0^2}\bar{J}_K(s) + \frac{mB_0}{F_0^2}\bar{J}_\pi(s) \right). \end{aligned} \tag{A.12}$$

The last corrective term is a polynomial, linear in s. This term picks up contributions from four coupling constants of the $O(p^6)$ chiral lagrangian, C_{39}^r, C_{20}^r, C_{21}^r and C_{94}^r in the notation of [10]. The part proportional to s is as follows:

$$\begin{aligned} A = 64C_{39}^r + \frac{1}{\pi^2} \Big[& \frac{3}{512\pi^2} L_K(L_\pi + L_K + L_\eta) + L_\pi \left(\frac{3}{512\pi^2} - \frac{3}{2}L_4^r \right) + \\ & + L_K \left(\frac{3}{128\pi^2} - 3L_4^r - L_5^r \right) + L_\eta \left(\frac{3}{512\pi^2} - \frac{5}{6}L_4^r - \frac{2}{9}L_5^r \right) + \\ & + \frac{9}{512\pi^2} - \frac{16}{3}L_4^r - \frac{11}{9}L_5^r \Big]. \end{aligned} \tag{A.13}$$

The constant terms, finally are

$$\begin{aligned} B = & \frac{mB_0}{F_0^2} \Big[32(8C_{20}^r + 48C_{21}^r + C_{94}) - \frac{1}{864\pi^4} L_\eta(-9L_\pi + 12L_K - L_\eta) + \\ & \quad + \frac{L_\pi}{\pi^2}\left(12S_6 + \frac{1}{96\pi^2} \right) + \frac{L_K}{\pi^2}\left(20S_6 + 6S_8 - \frac{1}{96\pi^2} \right) + \\ & \quad + \frac{L_\eta}{\pi^2}\left(4S_6 + \frac{8}{9}S_8 - \frac{11}{5184\pi^2} \right) + \\ & \quad + \frac{1}{\pi^2}\left(\frac{206}{9}S_6 - \frac{16}{27}S_7 + \frac{124}{27}S_8 + \frac{1}{5184\pi^2} \right) \Big] + \\ & + \frac{m_s B_0}{F_0^2} \Big[256(C_{20}^r + 3C_{21}^r) - \frac{1}{288\pi^4} L_\eta(-L_\eta + 5L_K) + \\ & \quad + \frac{L_K}{\pi^2}\left(16S_6 + 6S_8 - \frac{1}{96\pi^2} \right) + \frac{L_\eta}{9\pi^2}\left(48S_6 + 16S_7 + 16S_8 - \frac{37}{576\pi^2} \right) + \\ & \quad + \frac{1}{\pi^2}\left(\frac{118}{9}S_6 + \frac{16}{27}S_7 + \frac{140}{27}S_8 - \frac{19}{5184\pi^2} \right) \Big], \end{aligned} \tag{A.14}$$

and

$$C = -\frac{1}{16\pi^2}\left(2\,\frac{\Delta M_K^2}{M_K^2} + \frac{4}{9}\,\frac{\Delta M_\eta^2}{M_\eta^2} \right). \tag{A.15}$$

The terms proportional to the mass corrections ΔM_K^2 and ΔM_η^2 (see eqs. (A.6) and (A.15)) may eventually be absorbed into the $O(p^4)$ expression for $\Pi_6(s)$, which amounts to replace the lowest order expression for the pseudo-scalar masses there by their expression up to $O(p^4)$. Formula (2.23) in the text is obtained from $B + C$, setting $m = 0$.

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